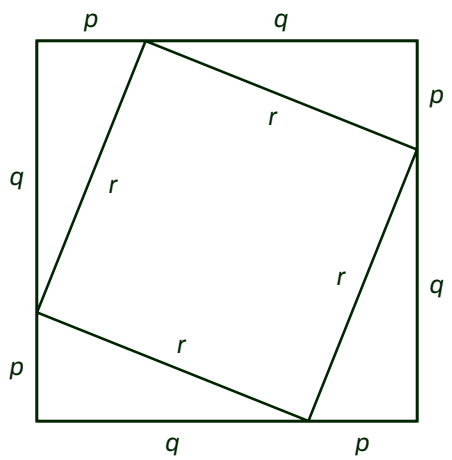
GRADE 8 · SPATIAL AND GEOMETRIC REASONING

Explain a proof of the Pythagorean theorem

NAME	DATE	SCORE / 10

1 The large square below has sides p + q. Inside it are four copies of one right triangle, with legs p and q and hypotenuse r. Use the diagram to explain, step by step, why $p^2 + q^2 = r^2$.



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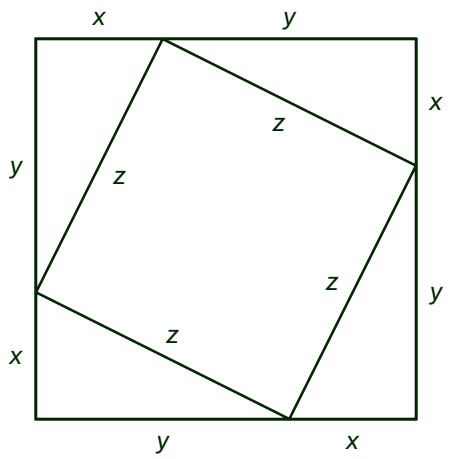

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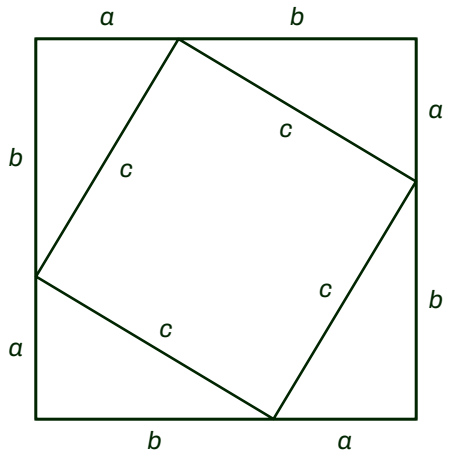

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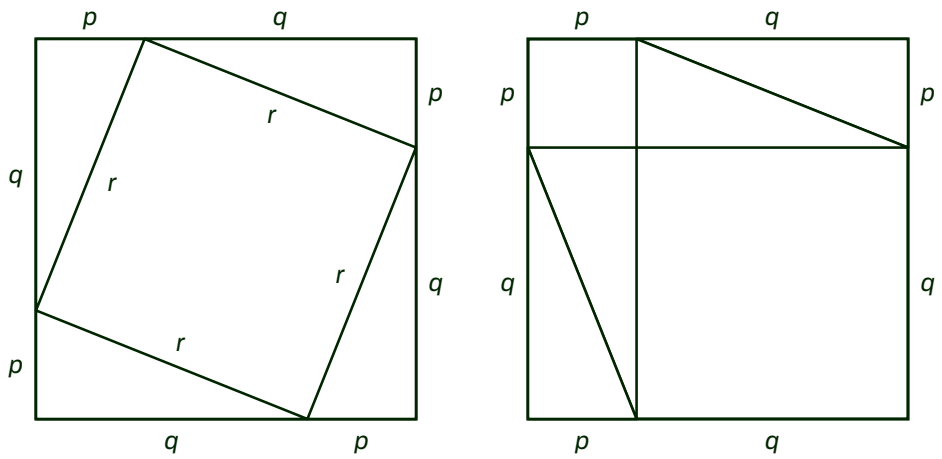

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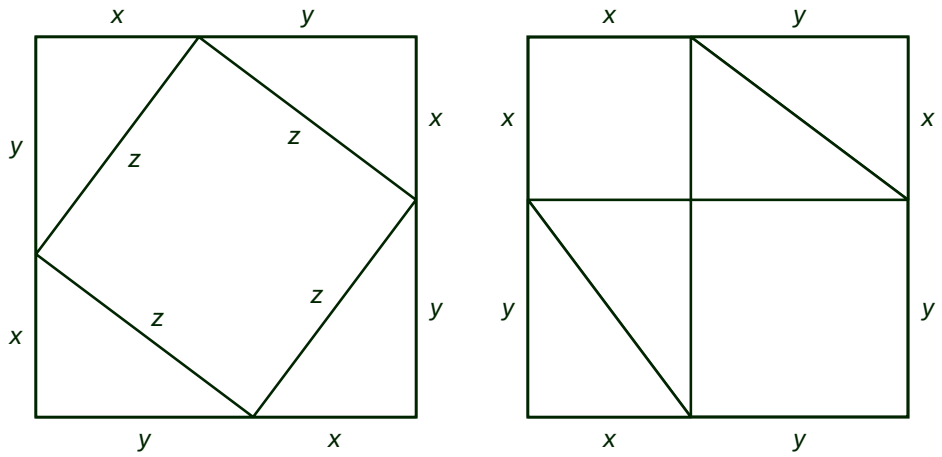

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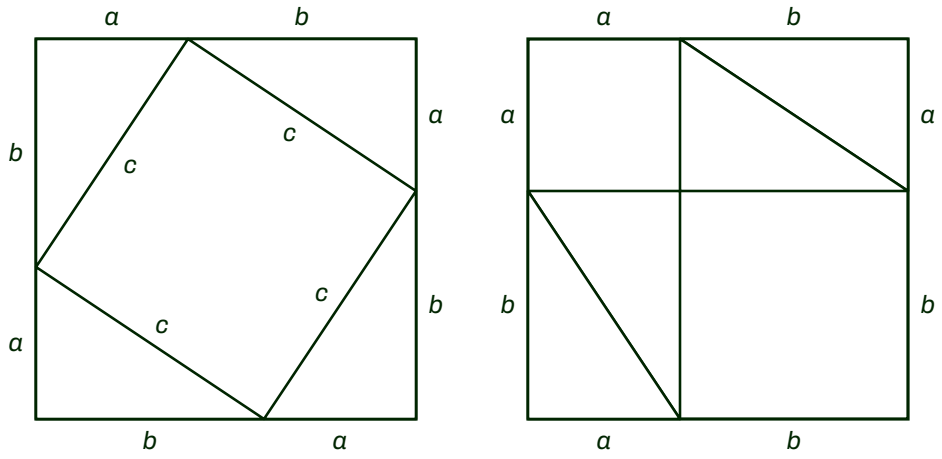

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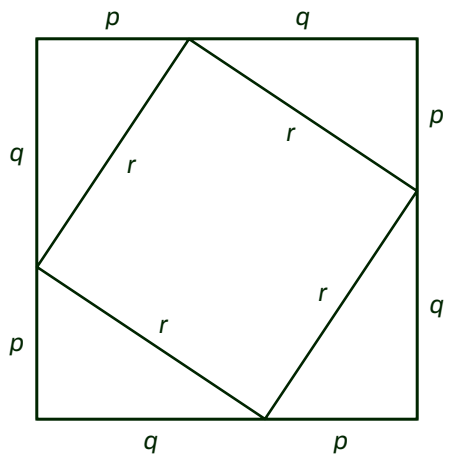

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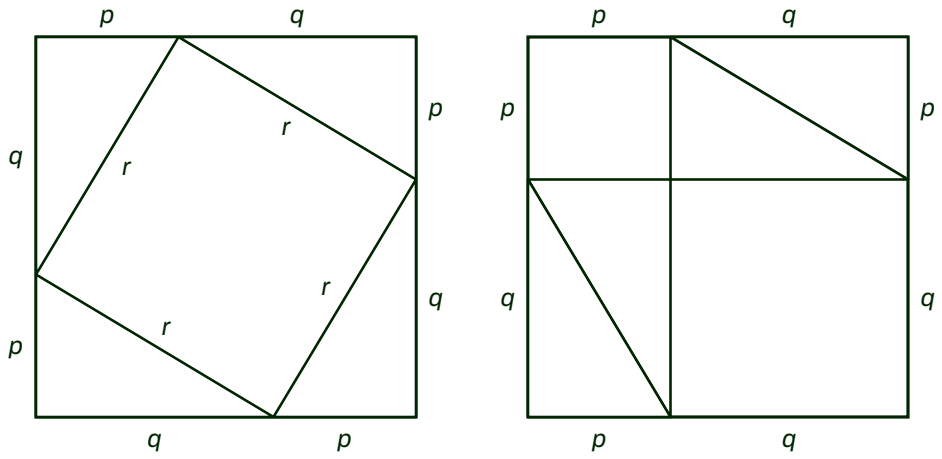

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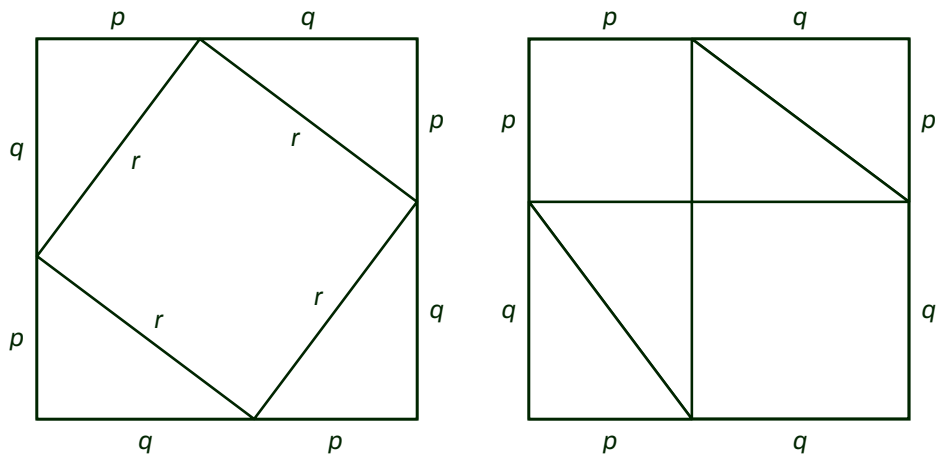

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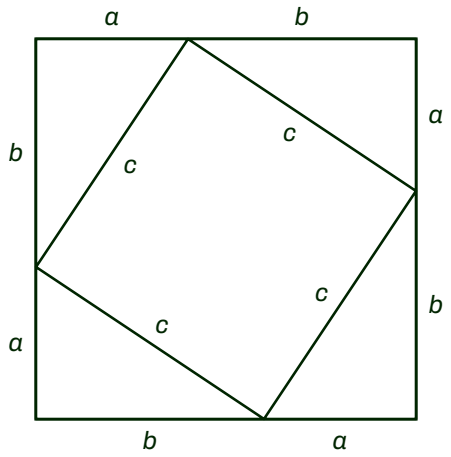

ANSWER KEY · GRADE 8 · SPATIAL AND GEOMETRIC REASONING

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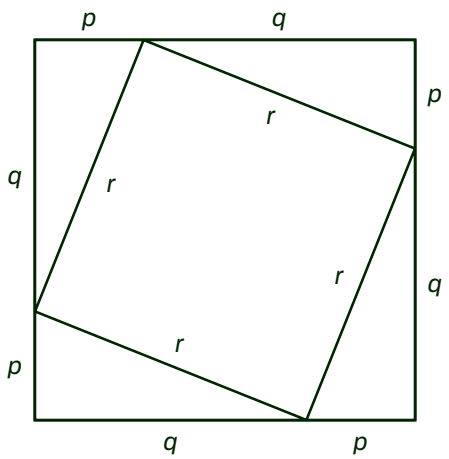
1 A written explanation



A model answer: The large square has sides p + q, so its area is $(p + q)^2 = p^2 + 2pq + q^2$. The same square is made of the four triangles and the square in the middle. Each triangle has area ½pq, so the four make 2pq; the middle square has sides r, so its area is r^2. So $p^2 + 2pq + q^2 = 2pq + r^2$, and taking 2pq from both sides leaves $p^2 + q^2 = r^2$. (The middle shape is a square: its sides are all r, and at each of its corners the two smaller angles of a triangle, which add to 90°, leave 90°.)

What counts as right: Counts as right if it makes these points: (1) the large square's area is $(p + q)^2$, which is $p^2 + 2pq + q^2$; (2) the same area is the four triangles, 4 × ½pq = 2pq, plus the middle square, r^2; (3) setting the two equal and taking away 2pq leaves $p^2 + q^2 = r^2$. A stronger answer also says why the middle shape is a square. Checking one triangle with numbers (3, 4 and 5) does not count: the reason must work for every right triangle.

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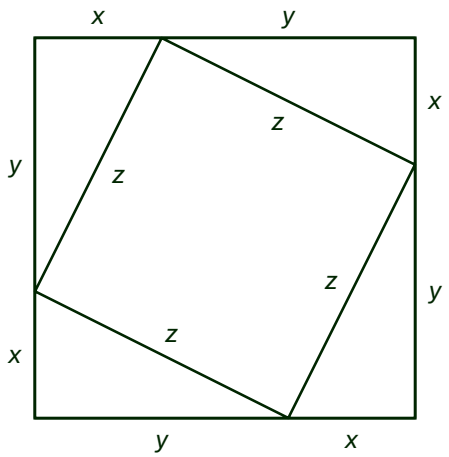

A model answer: The large square has sides x + y, so its area is $(x + y)^2 = x^2 + 2xy + y^2$. The same square is made of the four triangles and the square in the middle. Each triangle has area ½xy, so the four make 2xy; the middle square has sides z, so its area is z^2. So $x^2 + 2xy + y^2 = 2xy + z^2$, and taking 2xy from both sides leaves $x^2 + y^2 = z^2$. (The middle shape is a square: its sides are all z, and at each of its corners the two smaller angles of a triangle, which add to 90°, leave 90°.)

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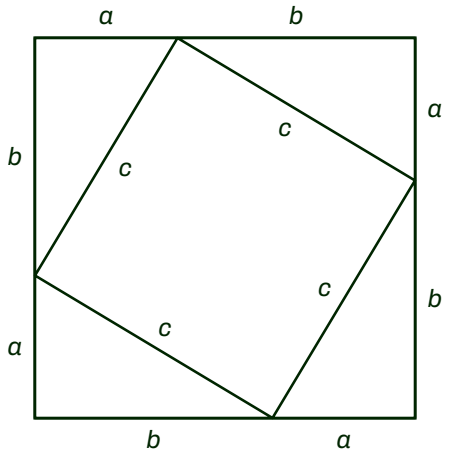

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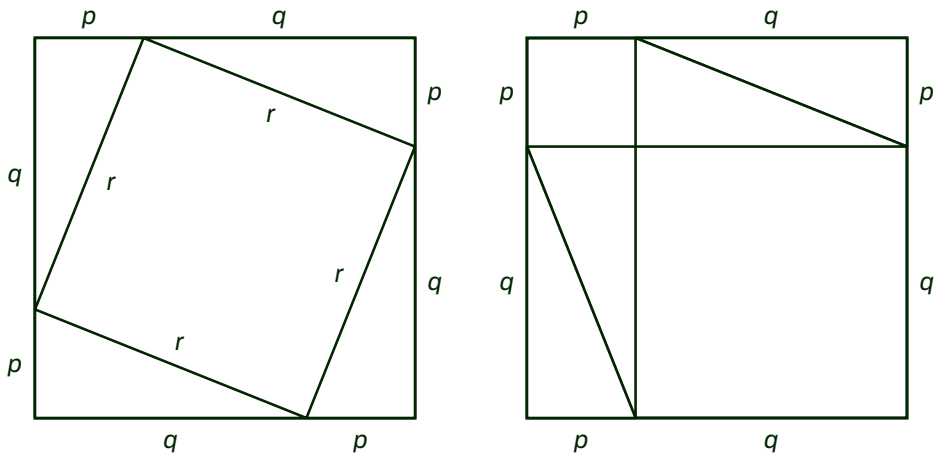

A model answer: Both large squares have sides p + q, so they have the same area. Each one holds the same four right triangles. Take the four triangles out of each square, and the area left over must be the same. In the first square what is left is one square with sides r, area r^2. In the second it is two squares, with sides p and q, area $p^2 + q^2$. So $p^2 + q^2 = r^2$.

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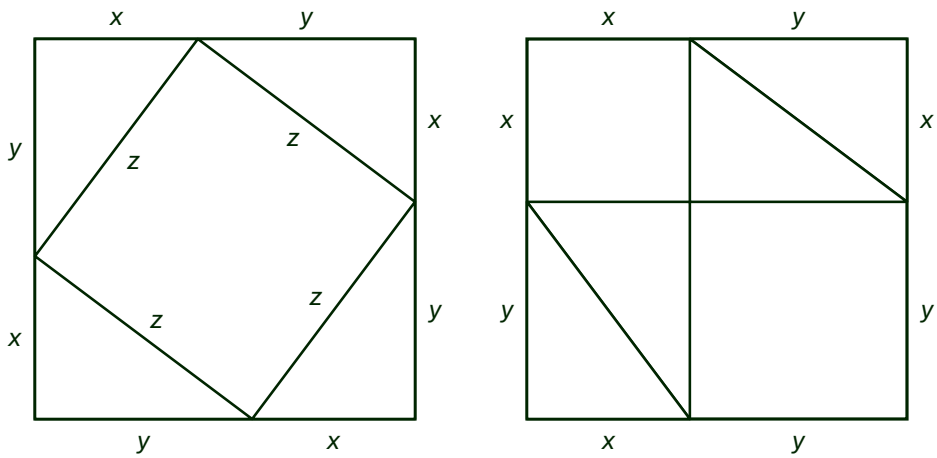

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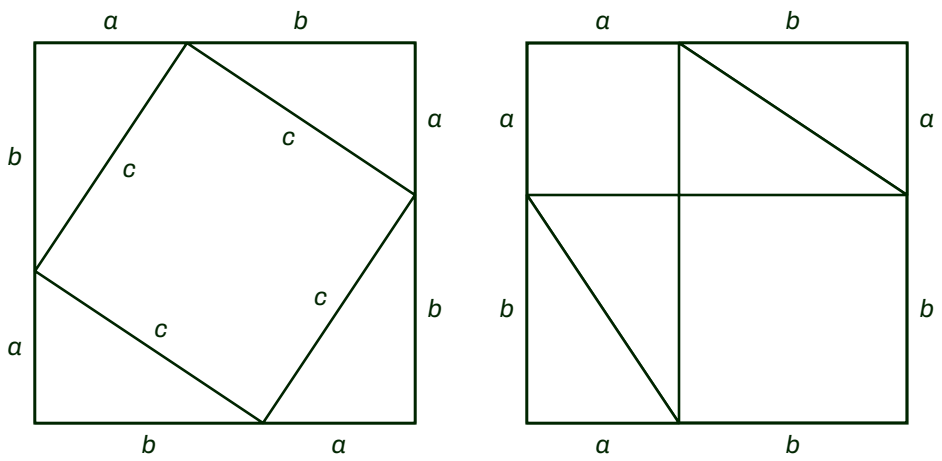

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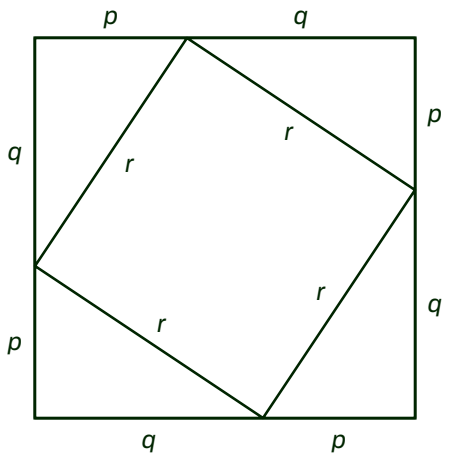

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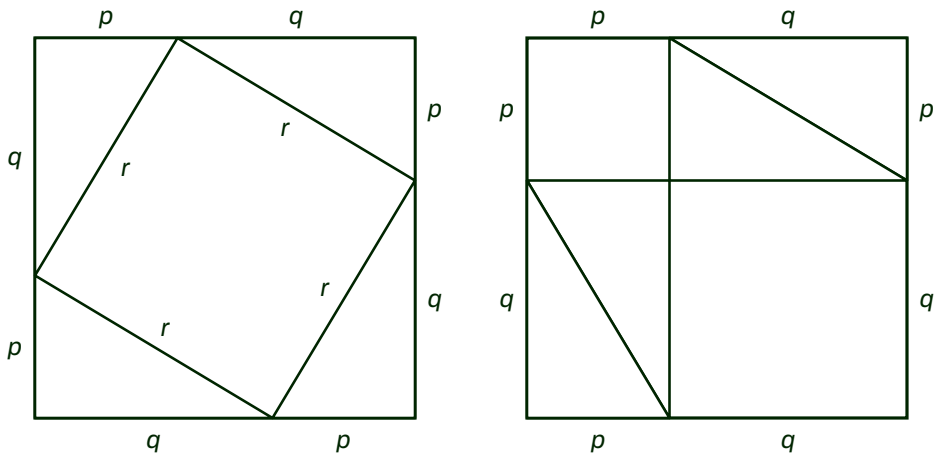

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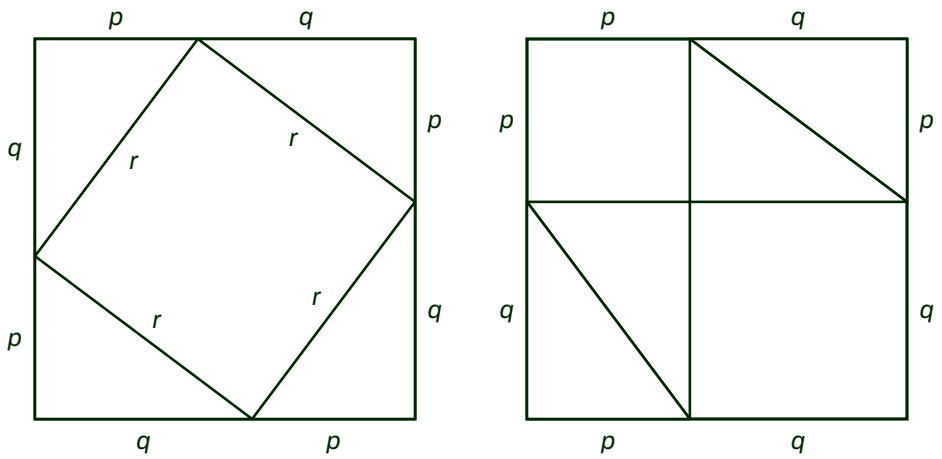

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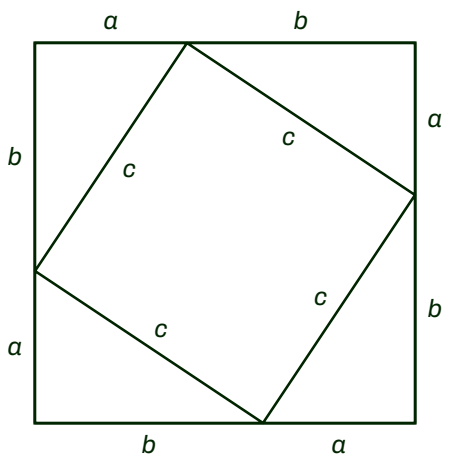

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